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An Inversion for a Jacobi Integral Transformation

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Ta Li [1], Buschman [2,3], and the author [4] have given inversion formulas for integral transforms involving Chebyshev, Legendre, and Gegenbauer polynomials. Since all of these polynomials are special cases of the Jacobi polynomial, it seems likely that a similar inversion should hold for the Jacobi polynomial. We establish such an inversion pair. Although the argument of the Jacobi polynomial is not as simple as that appearing in the other transforms, this seems to be unavoidable because the Jacobi polynomial is represented by a hypergeometric function which does not permit a quadratic transformation. Although this Jacobi transform does reduce to Chebyshev, Legendre, and Gegenbauer transforms with the same argument, the reduction to a Gegenbauer transform with an argument which is the same as that given in [3,4] is not given here because, even though the reduction is straightforward, it is quite tedious.

The form of the inversion given here is in the simple but unsymmetric form which was not given by Ta Li or Buschman and was mentioned in [4] as a special example of the more general symmetric case. A similar simple inversion formula for the Legendre function transform was given by Erdelyi [5]. Although the Chebyshev, Legendre, and Gegenbauer polynomials are special cases of both the Legendre function and the Jacobi polynomial, neither of the last two follows from the other. The general symmetric and simple inversions for all of these transforms follow directly from the hypergeometric transform of the author [6], but the simple transform given here requires only the most elementary fractional integral properties rather than the generalized ones used in [5] or the different generalization used in [6].

We define $(\frac{d}{dx})^{\alpha}\ g(x)$ to be the ordinary derivative of $g(x)$, $\alpha = 0,1,2,\ldots$ and to be the fractional derivative if α is not an integer, defined as in [7]

$$(\frac{d}{dx})^{\alpha}\ g(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^{n} \int_{x}^{\infty} g(s)(s-x)^{n-\alpha-1}ds \qquad n-1 < \alpha < n$$

(1) and $\qquad n = 0,1,2,\ldots$

$$(\frac{d}{dx})^{\alpha}(\frac{d}{dx})^{\beta}\ g(x) = (\frac{d}{dx})^{\alpha+\beta}\ g(x). \qquad \alpha > -1$$

Then provided $F(\sigma)$ and its first $[n+\alpha+\beta+1]$ derivatives ($[\delta]$ is the integral part of δ) are continuous and if $F(\sigma)$ and its first $[n+\alpha+\beta+1]$ derivatives vanish for $\sigma \geq 1$, we have the

Theorem: If

$$(2) \quad F(\sigma) = \int_{\sigma}^{1} (s-\sigma)^{\alpha}\ P_{n}^{\alpha,\beta}(\frac{2s}{\sigma})\ Y(s)ds$$

then

$$(3) \quad Y(u) = \frac{(-1)^{[n+\alpha+\beta+1]}\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta)} \int_{u}^{1} v^{-n}(v-u)^{n+\beta-1}\{\frac{d}{dv}\}^{n+\alpha+\beta+1}[v^{n}F(v)]dv.$$

Proof:

We consider the integral obtained by substituting equation (3)

into (2).

$$F(\sigma) = \frac{(-1)^{[n+\alpha+\beta+1]}\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta)} \int_\sigma^1 (s-\sigma)^\alpha P_n^{\alpha,\beta}(\frac{2s}{\sigma}-1) \times$$

(4)

$$\times \int_s^1 v^{-n}(v-s)^{n+\beta-1}\{\frac{d}{dv}\}^{n+\alpha+\beta+1}[v^n F(v)]dvds$$

$$= \frac{(-1)^{[n+\alpha+\beta+1]}\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta)} \int_\sigma^1 v^{-n}\{\frac{d}{dv}\}^{n+\alpha+\beta+1}[v^n F(v)] \times$$

(5)

$$\times \int_\sigma^v (s-\sigma)^\alpha (v-s)^{n+\beta-1} P_n^{\alpha,\beta}(\frac{2s}{\sigma}-1)dsdv$$

(6) $$= \frac{(-1)^{[n+\alpha+\beta+1]}}{\Gamma(n+\beta)} \int_\sigma^1 v^{-n}\{\frac{d}{dv}\}^{n+\alpha+\beta+1}[v^n F(v)] \cdot I(v)dv$$

where

(7) $$I(v) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \int_\sigma^v (s-\sigma)^\alpha (v-s)^{n+\beta-1} P_n^{\alpha,\beta}(\frac{2s}{\sigma}-1)ds.$$

Now we use the relation between the Jacobi polynomial and the hypergeometric function [8, vol. 2].

(8) $$P_n^{\alpha,\beta}(\frac{2s}{\sigma}-1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} F(-n, n+\alpha+\beta+1; \alpha+1; 1-\frac{s}{\sigma})$$

and the definition of the hypergeometric function [8, vol. 1]

$$F(-n, b; c; x) = \sum_{k=0}^{n-1} \frac{(-n)_k (b)_k}{(c)_k k!} x^k$$

in equation (7) to get

(9) $$I(v) = \frac{1}{\Gamma(n+\alpha+\beta+1)} \sum_{k=0}^{n-1} \frac{(-1)^k(-n)_k \Gamma(\alpha+\beta+n+k+1)}{\sigma^k k! \Gamma(\alpha+k+1)} \cdot J$$

where

$$(10) \qquad J = \int_{\sigma}^{v} (s - \sigma)^{\alpha+k}(v - s)^{n+\beta-1} ds.$$

J is a Euler integral and its value can be written as

$$\frac{\Gamma(\alpha + k + 1)\Gamma(n + \beta)}{\Gamma(\alpha + \beta + n + k + 1)} (v - \sigma)^{\alpha+\beta+n+k}$$

so that

$$(11) \quad I(v) = \frac{\Gamma(n + \beta)(v - \sigma)^{\alpha+\beta+n}}{\Gamma(n + \alpha + \beta + 1)} \sum_{k=0}^{n-1} \frac{(-n)_k (1 - \frac{v}{\sigma})^k}{k!}.$$

We note that the sum can be evaluated using

$$(12) \qquad F(-n,1;1;1 - \tfrac{v}{\sigma}) = (\tfrac{v}{\sigma})^n.$$

This gives

$$(13) \qquad I(v) = \frac{\Gamma(n + \beta)(v - \sigma)^{\alpha+\beta+n} v^n}{\Gamma(n + \alpha + \beta + 1)\ \sigma^n}$$

and it follows that

$$(14) \quad F(\sigma) = \frac{(-1)^{[n+\alpha+\beta+1]}}{\Gamma(n + \alpha + \beta + 1)\sigma^n} \int_{\sigma}^{1} (v - \sigma)^{\alpha+\beta+n} (\tfrac{d}{dv})^{n+\alpha+\beta+1} [v^n F(v)] dv.$$

Let

$$(15) \quad n + \alpha + \beta + 1 = [n + \alpha + \beta + 1] + \gamma = p + \gamma \qquad p = 0,1,2,\ldots$$

and then

$$(16) \quad F(\sigma) = \frac{(-1)^p}{\Gamma(p + \gamma)\sigma^n} \int_{\sigma}^{1} (v - \sigma)^{p+\gamma-1} \{\tfrac{d}{dv}\}^p [(\tfrac{d}{dv})^{\gamma} [v^n F(v)]] dv.$$

We integrate by parts p times to get

(17) $$F(\sigma) = \frac{1}{\Gamma(\gamma)\sigma^n}\int_\sigma^1 (v - \sigma)^{\gamma-1}\{\frac{d}{dv}\}^\gamma[v^n F(v)]dv.$$

But by definition of the fractional derivative (1) for a function which is zero for $\sigma \geq 1$,

(18) $$\frac{1}{\Gamma(\gamma)}\int_\sigma^1 (v - \sigma)^{\gamma-1}G(v)dv = (\frac{d}{d\sigma})^{-\gamma}G(\sigma)$$

so

(19) $$F(\sigma) = \frac{1}{\sigma^n}(\frac{d}{d\sigma})^{-\gamma}(\frac{d}{d\sigma})^\gamma[\sigma^n G(\sigma)] = F(\sigma),$$

which proves the theorem.

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